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Noncommutative Geometry Inspired Rotating Black Hole in Three Dimensions

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Abstract. We find a new rotating black hole in three-dimensional anti-de Sitter space using an anisotropic perfect fluid inspired by the noncommutative black hole. We deduce the thermodynamical quantities of this black hole and compare them with those of a rotating BTZ solution.

PACS. 04.70.Dy black holes thermodynamics – 97.60.Lf Black holes

1 Introduction

The theoretical discovery of radiating black holes [1] disclosed the first physically relevant window on the mysteries of quantum gravity. After many years of intensive research in this field various aspects of the problem still remain under debate. For instance, a fully satisfactory description of the late stage of black hole evaporation is still missing. The string/black hole correspondence principle [2] suggests that in this extreme regime stringy effects cannot be neglected. This is just one of many examples of how the development of string theory has affected various aspects of theoretical physics. Among different outcomes of string theory, we focus on the result that target spacetime coordinates become noncommuting operators on a D-brane [3]. Thus, string-brane coupling has put in evidence the necessity of spacetime quantization.

The noncommutativity of spacetime can be encoded in the commutator

$$[x^{\mu}, x^{\nu}] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is an anti-symmetric matrix which determines the fundamental cell discretization of spacetime much in the same way as the Planck constant $\hbar$ discretizes the phase space. This noncommutativity provides a black hole with a minimum scale $\sqrt{\theta}$ known as the noncommutative black hole [4,5,6,7,8,9], whose commutative limit is the Schwarz-schild metric. Myung and Kim, [10] have studied the thermodynamics and evaporation process of this noncommutative black hole while the entropy issue of this black hole was discussed in [11,12] and Hawking radiation was considered in [13].

In this paper, we construct a new rotating black hole in AdS3 spacetime using an anisotropic perfect fluid inspired by the 4D noncommutative black hole, resulting in a solution with two horizons. We compare the thermodynamics of this black hole with that of a rotating BTZ black hole [14,15].

2 Derivation of the Rotating Solution

It has been shown [4,5] that the noncommutativity eliminates point-like structures in favor of smeared objects in flat space-time. A way of implementing the effect of smearing is a substitution rule: in four-dimensional (4D) spacetimes, the Dirac delta function $\delta^{4D}(r)$ is replaced by a Gaussian distribution with minimal width $\sqrt{\theta}$ [4,5,6,7,8,9],

$$\rho^{4D}(r) = \frac{M}{(4\pi\theta)^{3/2}} e^{-r^2/4\theta} \tag{2}$$

and the corresponding mass distribution is given by

$$m^{4D}(r) = 4\pi \int_0^r r'^2 \rho^{4D}(r')\, dr' = \frac{2M}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right), \tag{3}$$

where $\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)$ is the lower incomplete gamma function defined as

$$\gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt. \tag{4}$$

In three dimensions, the Dirac delta function $\delta^{3D}(r)$ is replaced by a Gaussian distribution with minimal width $\sqrt{\theta}$, [16],

$$\rho^{3D}(r) = \frac{M}{4\pi\theta} e^{-r^2/4\theta} \tag{5}$$

and the corresponding mass distribution is now

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$$m^{3D}(r) = 2\pi \int_0^r r' \rho^{3D}(r')\, dr' = M\gamma\left(1, \frac{r^2}{4\theta}\right) \quad (6)$$
$$= M\left(1 - e^{-r^2/4\theta}\right). \quad (7)$$

In order to find a black hole solution in AdS_3 spacetime, we introduce the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} + \frac{1}{\ell^2} g_{\mu\nu} \quad (8)$$

where ℓ is related with the cosmological constant by

$$\Lambda = -\frac{1}{\ell^2}. \quad (9)$$

The energy-momentum tensor will take the anisotropic form

$$T^\mu_\nu = \mathrm{diag}\left(-\rho, p_r, p_\perp\right). \quad (10)$$

In order to completely define this tensor, we rely on the covariant conservation condition $T^{\mu\nu}{}_{,\nu} = 0$. This gives the source as an anisotropic fluid of density ρ, radial pressure

$$p_r = -\rho \quad (11)$$

and tangential pressure

$$p_\perp = -\rho - r\partial_r \rho. \quad (12)$$

Solving the above equations, we find the line element

$$ds^2 = -f(r)\, dt^2 + f^{-1}(r)\, dr^2 + r^2 \left(d\varphi + N^\varphi dt\right)^2, \quad (13)$$

where

$$f(r) = -8M\left(1 - e^{-r^2/4\theta}\right) + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \quad (14)$$
$$N^\varphi = -\frac{J}{2r^2}. \quad (15)$$

Note that when $\frac{r^2}{4\theta} \to \infty$, either for considering a large black hole ($r \to \infty$) or for considering the commutative limit ($\theta \to 0$), we obtain the well known BTZ rotating solution with angular momentum J and total mass M,

$$f^{BTZ}(r) = -8M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}. \quad (16)$$

The line element (13) describes the geometry of a noncommutative black hole with event horizons given by the condition

$$f(r_\pm) = -8M\left(1 - e^{-r_\pm^2/4\theta}\right) + \frac{r_\pm^2}{l^2} + \frac{J^2}{4r_\pm^2} = 0. \quad (17)$$

This equation cannot be solved in closed form. However, by plotting $f(r)$ one can read intersections with the r-axis and determine numerically the existence of horizon(s)

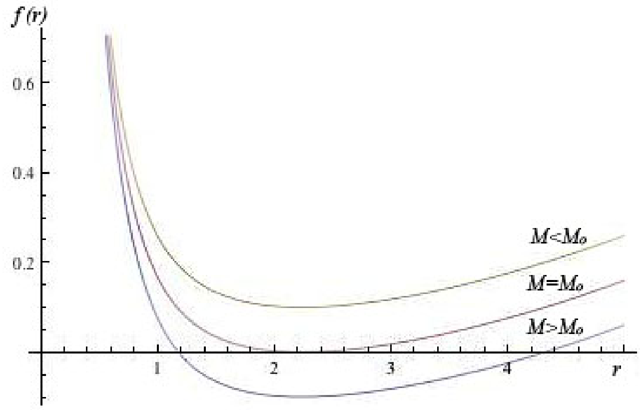

Fig. 1. Metric function f as a function of r. We have taken the values $\theta = 0.1$, $\ell = 10$ and $J = 1$. The minimum mass is $M_o \approx 0.0125$.

and their radii. Fig. 1 shows that the existence of angular momentum introduces new behavior with respect to the noncommutative black hole studied by Myung and Yoon [16] and others reported before [17, 18]. Instead of a single event horizon, there are different possibilities:

1. Two distinct horizons for $M > M_o$
2. One degenerate horizon (extremal black hole) for $M = M_o$
3. No horizon for $M < M_o$.

In view of this results, there can be no black hole if the original mass is less than the lower limit mass M_o. The horizon of the extremal black hole is determined by the conditions $f = 0$ and $\partial_r f = 0$, which gives

$$r_o^4 \left[\frac{1 - \left(1 + \frac{r_o^2}{4\theta}\right) e^{-r_o^2/4\theta}}{1 - \left(1 - \frac{r_o^2}{4\theta}\right) e^{-r_o^2/4\theta}}\right] = \frac{J^2 \ell^2}{4} \quad (18)$$

and then, the mass of the extremal black hole can be written as

$$M_o = \frac{\left(\frac{r_o^2}{\ell^2} + \frac{J^2}{4r_o^2}\right)}{8\left(1 - e^{-r_o^2/4\theta}\right)}. \quad (19)$$

In the commutative limit, $\theta \to 0$, the extreme black hole has the horizon at

$$r_o^{BTZ} = \sqrt{\frac{J\ell}{2}} \quad (20)$$

and its mass is

$$M_o^{BTZ} = \frac{1}{8}\frac{J}{\ell}.$$

3 Thermodynamics

The black hole temperature is given by

$$T_H = \frac{1}{4\pi}\, \partial_r f\big|_{r_+} \quad (21)$$

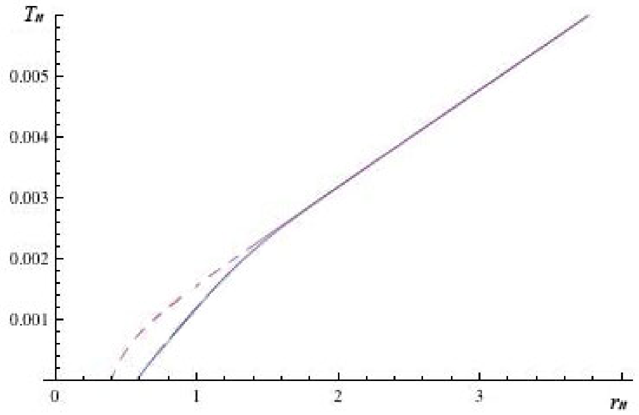

Fig. 2. Hawking temperature versus r_H. The solid line represents the temperature for the noncommutative black hole with $\theta = 0.1$. The dashed line represents the temperature for the rotating BTZ black hole. In both cases we have taken the values $\ell = 10$ and $J = 0.03$.

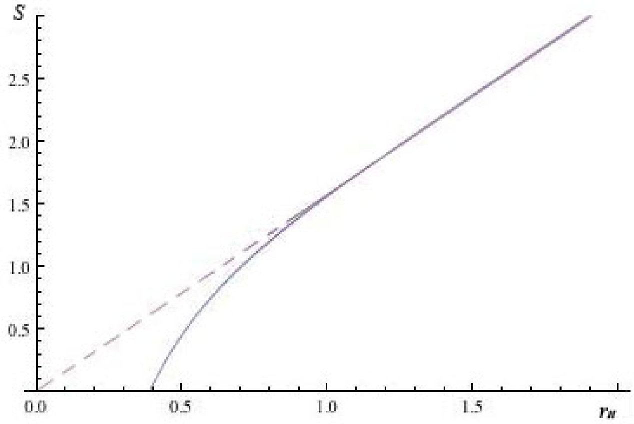

Fig. 3. Entropy versus r_H. The solid line represents the entropy of the noncommutative black hole with $\theta = 0.1$. The dashed line represents the entropy of the rotating BTZ black hole. In both cases we use $\ell = 10$.

$$T_H = \frac{r_+}{2\pi\ell^2}\left[1 - \frac{J^2\ell^2}{4r_+^4} + \frac{\left(r_+^2 + \frac{J^2\ell^2}{4r_+^2}\right)}{4\theta\left(1 - e^{r_+^2/4\theta}\right)}\right]. \quad (22)$$

For large black holes, i.e. $\frac{r_+^2}{4\theta} >> 0$, one recovers the temperature of the rotating BTZ black hole,

$$T_H^{BTZ} = \frac{r_+}{2\pi\ell^2}\left[1 - \frac{J^2\ell^2}{4r_+^4}\right]. \quad (23)$$

As shown in the Fig. 2., the temperature is a monotonically increasing function of the horizon radius for large black holes and the temperature of the extreme black hole is zero.

The first law of thermodynamics for a rotating black hole reads

$$dM = T_H dS + \Omega dJ, \quad (24)$$

where the angular velocity of the black hole is given by

$$\Omega = \left(\frac{\partial M}{\partial J}\right)_{r_+} = \frac{J}{2r_+^2}, \quad (25)$$

that is exactly the same of the rotating BTZ solution. We calculate the entropy as

$$S = \int_{r_o}^{r_+} \frac{1}{T_H} dM \quad (26)$$

which finally gives

$$S = \frac{\pi}{2}\int_{r_o}^{r_+} \left(\frac{1}{1 - e^{-\xi^2/4\theta}}\right) d\xi. \quad (27)$$

The entropy as a function of r_+ is depicted in Fig. 3. Note that, in the large black hole limit, the entropy function corresponds to the Bekenstein-Hawking entropy (area law), $S_{BH} = \frac{\pi r_+}{2}$, for the rotating BTZ geometry.

4 Conclusion

We construct a noncommutative rotating black hole in AdS_3 spacetime using an anisotropic perfect fluid inspired by the 4D noncommutative black hole. As well as its $4D$ counterpart, this black hole has two horizons. We compare the thermodynamics of this black hole with that of a rotating BTZ black hole. The Hawking temperature, angular velocity and entropy of large noncommutative black hole approach those of BTZ.

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